

# ON THE STABILITY OF LINEAR SYSTEMS WITH DELAYED STATE DEFINED OVER FINITE TIME INTERVAL

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## Abstract

Paper extends some basic results from the area of finite time and practical stability to linear, continuous, time invariant time-delay systems. Sufficient conditions of this kind of stability, for particular class of time-delay systems are derived.

## 1 Introduction

Weiss and Infante [1,2] have introduced various notations of stability over finite time interval for continuous-time systems and constant set trajectory bounds. Further development of these results were due to many other authors. In the context of finite time and practical stability, for the linear time delay system, various results were for the first time obtained by Nenadić *et al.* [3] and Debeljković *et al.* [4]. Further extensions of these results were due to Debeljković *et al.* [5] and Lazarević *et al.* [6].

## 2 Preliminaries

A linear, multivariable time-delay system can be represented by differential equation:

$$\dot{\mathbf{x}}(t) = A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t - \tau), \quad (1)$$

and with associated function of initial state:

$$\mathbf{x}(t) = \psi_{\mathbf{x}}(t), \quad -\tau \leq t \leq 0, \quad (2)$$

Equation (1) is referred to as homogenous or the unforced state equation,  $\mathbf{x}(t)$  is the state vector,  $A_0$  and  $A_1$  are constant system matrices of appropriate dimensions, and  $\tau$  is pure time delay,  $\tau = \text{const.}$  ( $\tau > 0$ ).

Dynamical behavior of autonomous system (1) is defined over time interval  $J = \{t_0, t_0 + T\}$ , where quantity  $T$  may be either a positive real number or symbol  $+\infty$ , so finite time stability and practical stability can be treated simultaneously. It is obvious that  $J \in \mathbb{R}$ .

Let index  $\beta$  stands for the set of all allowable states of system and index  $\alpha$  for the set of all initial states of the system, such that the set  $S_\alpha \subseteq S_\beta$ . In general, one may write:

$$S_\rho = \{\mathbf{x} : \|\mathbf{x}\|_Q^2 < \rho\}, \quad (3)$$

where  $Q$  will be assumed to be symmetric, positive-definite, real matrix.

Let  $\|\cdot\|_i$  be any vector norm (e.g.,  $i = 1, 2, \infty$ ) and  $\|(\cdot)\|$  the matrix norm induced by this vector. For our purpose, we use  $\|\cdot\|_2 = (\mathbf{x}^T \mathbf{x})^{1/2}$  and  $\|(\cdot)\|_2 = \lambda_{\max}^{1/2}(A^* A)$ .

The matrix measure  $\mu$  for any matrix  $A \in \mathbb{C}^{n \times n}$  is defined as follows

$$\mu(A) \stackrel{\Delta}{=} \lim_{\varepsilon \rightarrow 0} \frac{\|I + \varepsilon A\| - 1}{\varepsilon}. \quad (4)$$

For our purpose we use:

$$\mu_2(A) = \frac{1}{2} \max_i \lambda_i(A^* + A), \quad (5)$$

## 3 Previous results

**Definition 1:** System given by (1) satisfying initial condition (2) is finite time stable w.r.t.  $[\zeta(t), \beta, T]$  if and only if:

$$\psi_{\mathbf{x}}^T(t) \psi_{\mathbf{x}}(t) < \zeta(t), \quad (6)$$

implies:

$$\mathbf{x}(t)^T \mathbf{x}(t) < \beta, \quad (7)$$

$\zeta(t)$  being scalar function with the property  $0 < \zeta(t) \leq \alpha$ ,  $-\tau \leq t \leq 0$ , where  $\alpha$  is a real positive number and  $\beta \in \mathbb{R}$ ,  $\beta > \alpha$ .

**Theorem 1:** The system given by (1) with the initial function (2) is finite time stable with respect to  $\{\alpha, \beta, \tau, T\}$ , if the following condition is satisfied:

$$\|\Phi\| < \frac{\sqrt{\beta/\alpha}}{1 + \tau \|A_1\|_2}, \quad \forall t \in [0, T] \quad (8)$$

where  $\|(\cdot)\|$  is Euclidean norm and  $\Phi(t)$  is fundamental matrix of system (1), Nenadić *et al.* [3], Debeljković *et al.* [4].

**Theorem 2.** The system given by (1) with initial function (2) is finite time stable w.r.t.  $\{\alpha, \beta, \tau, T\}$  if the following condition is satisfied

$$e^{\mu_2(A_0)t} < \frac{\sqrt{\beta/\alpha}}{1 + \tau \|A_1\|_2}, \quad \forall t \in [0, T] \quad (9)$$

where  $\|(\cdot)\|$  denotes Euclidean norm, Debeljković *et al.* [5]. For the sake of brevity proofs are here omitted.

#### 4 Main results

**Definition 2:** System given by (1) satisfying initial condition (2) is finite time stable w.r.t.  $[\zeta(t), \beta, T]$  if and only if:

$$\|\psi_x(t)\|_2 < \zeta(t), \quad (10)$$

implies:

$$\|x(t)\|_2 < \beta, \quad \forall t \in J, \quad (11)$$

$\zeta(t)$  being scalar function with the property  $0 < \zeta(t) \leq \alpha$ ,  $-\tau \leq t \leq 0$ , where  $\alpha$  is a real positive number and  $\beta \in R, \beta > \alpha$ .

**Theorem 3:** The system given by (1) with the initial function (2) is finite time stable with respect to  $\{\alpha, \beta, \tau, T, \mu_2(A_0) \neq 0\}$ , if the following condition is satisfied:

$$e^{\mu_2(A_0)T} < \frac{\beta/\alpha}{1 + \mu_2^{-1}(A_0) \|A_1\|_2 (1 - e^{-\mu_2(A_0)\tau})}, \quad \forall t \in [0, T] \quad (12)$$

where  $\|(\cdot)\|_2$  denotes Euclidean norm.

**Proof.** The solution of (1) with initial function (2) can be expressed in terms of fundamental matrix as it is written below:

$$x(t) = \Phi(t)\psi_x(0) + \int_{-\tau}^0 \Phi(t-\theta-\tau)A_1\psi_x(\theta)d\theta. \quad (13)$$

Evaluating the norm  $\|(\cdot)\|_2$  of both sides of this equations yields

$$\|x(t)\|_2 \leq \|\Phi(t)\|_2 \|\psi_x(0)\|_2 + \int_{-\tau}^0 \|\Phi(t-\theta-\tau)\|_2 \|A_1\| \|\psi_x(\theta)\|_2 d\theta \quad (14)$$

Using the known inequality,

$$\|\exp(At)\|_2 \leq \exp\{\mu(A)t\}, \quad t \geq 0 \quad (15)$$

we have

$$\|x(t)\|_2 \leq e^{\mu_2(A_0)t} \|\psi_x(0)\|_2 + \int_{-\tau}^0 e^{\mu_2(A_0)(t-\theta-\tau)} \|A_1\|_2 \|\psi_x(\theta)\|_2 d\theta, \quad (16)$$

or, using (10)

$$\|x(t)\|_2 \leq \alpha e^{\mu_2(A_0)t} \left( 1 + \|A_1\|_2 \int_{-\tau}^0 e^{-\mu_2(A_0)(\theta+\tau)} d\theta \right), \quad (17)$$

and finally:

$$\|x(t)\|_2 \leq \alpha e^{\mu_2(A_0)t} \left\{ 1 + \mu_2^{-1}(A_0) \|A_1\|_2 (1 - e^{-\mu_2(A_0)\tau}) \right\}. \quad (18)$$

Applying the basic conditions of Theorem, equation (12), on the preceding inequality, one can get

$$\|x(t)\|_2 < \beta, \quad \forall t \in [0, T], \quad (19)$$

what has to be proved. When  $\tau=0$  or  $\|A_1\|_2=0$ , the problem is reduced to the case of the ordinary linear, differential systems. In particular case, when  $\mu_2(A_0)=0$ , we can give the following result.

**Theorem 4.** The system given by (1) with initial function (2) is finite time stable w.r.t.  $\{\alpha, \beta, \tau, T, \mu_2(A_0)=0\}$  if the following condition is satisfied

$$1 + \tau \|A_1\|_2 < \beta/\alpha. \quad (20)$$

**Proof.** Follows directly from Theorem 3 when  $\mu_2(A_0)=0$ .

#### 5. Conclusion

In this paper a new criterion for non-Lyapunov stability of particular linear time delay systems is derived. It is delay-dependent criterion and is expressed by a simple inequalities. This result represents further generalization of known results derived earlier and provide condition that is more applicable than those given in current literature, derived by the same authors.

#### 6. References

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